Discrete Mathematics Recitation Course Lecture 8

2013.06.06

張玟翔

8-1

Relations and Their Properties

8-1 Ex.8

- Give an example of a relation on a set that is
- a) symmetric and antisymmetric.
- b) neither symmetric nor antisymmetric.

- $\{(a,a),(b,b)\}$ on $\{a,b,c\}$.
- $\{(a,b),(b,a),(a,c)\}$ on $\{a,b,c\}$.
- antisymmetric $\neq$ not symmetric (asymmetric).

8-1 Ex.13

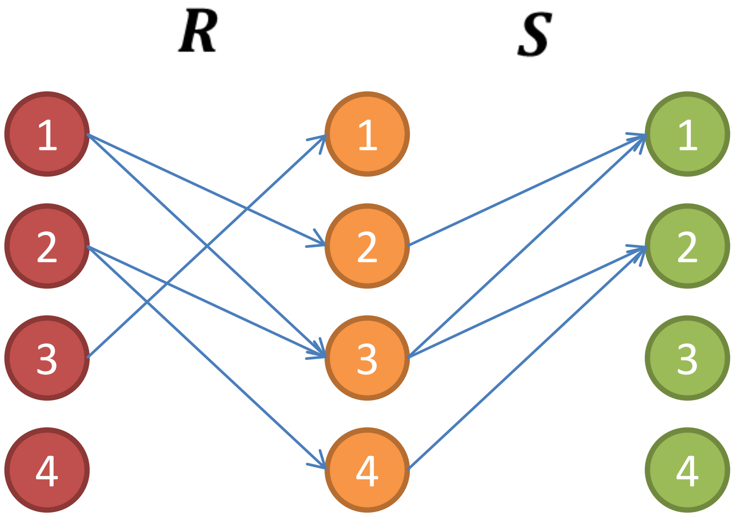
- Can a relation on a set be neither reflexive nor irreflexive?

- Yes, for instance $\{(1,1)\}$ on $\{1,2\}$.
- irreflexive $\neq$ not reflexive.

8-1 Ex.30

- Let R be the relation $\{(1,2),(1,3),(2,3),(2,4),(3,1)\}$, and let S be the relation $\{(2,1),(3,1),(3,2),(4,2)\}$.

Find $S \circ R$.



$S \circ R$
$= \{(1,1),(1,2),(2,1),(2,2)\}$

8-1 Ex.34

- $R_1 = \{(a,b) \in \mathbf{R}^2 | a > b\}$,
- $R_2 = \{(a,b) \in \mathbf{R}^2 | a \geq b\}$,
- $R_3 = \{(a,b) \in \mathbf{R}^2 | a < b\}$,
- Find
- b) $R_1 \circ R_2$
- c) $R_1 \circ R_3$

- For (a, c) to be in $R_1 \circ R_2$, $a \geq b$ and $b > c$.
- So $a > c$, and $R_1 \circ R_2 = R_1$.
- For (a, c) to be in $R_1 \circ R_3$, $a < b$ and $b > c$.
- This can be always done, so $R_1 \circ R_3 = \mathbb{R}^2$.

How many relations…?

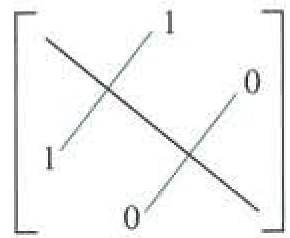
- How many relations are there on a set with n elements?
  - A relation on a set A is a subset of $A \times A$
  - $|A \times A| = n^2$
  - The number of subsets of a set with n^2 elements: 2^{n^2}
- How many reflexive relations are there on a set with n elements?
- Think about $\begin{bmatrix} 1 & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & 1 \end{bmatrix}_{n \times n}$ $\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}$
- 2^{n^2-n}

8-1 Ex.45

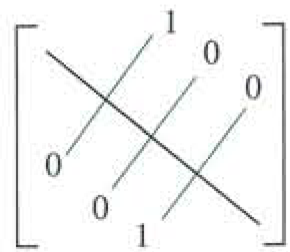
- How many relations are there on a set with *n* elements that are
- a) symmetric
- b) antisymmetric
- c) asymmetric
- d) irreflexive
- e) reflexive and symmetric
- f) neither reflexive nor irreflexive

8-1 Ex.45

- How many relations are there on a set with *n* elements that are
- a) symmetric
  - $2^n \cdot 2^{n(n-1)/2}$



- b) anti-symmetric
  - $2^n \cdot 3^{n(n-1)/2}$



- c) asymmetric
  - $3^{n(n-1)/2}$

8-1 Ex.45 (cont'd)

- d) irreflexive
  - 2^{n^2-n}
- e) reflexive and symmetric
  - $2^{n(n-1)/2}$
- f) neither reflexive nor irreflexive
  - $2^{n^2} - 2 \cdot 2^{n^2-n}$

8-2

n-ary Relations

8-2 Ex.2

- Which 4-tuples are in the relation $\{(a, b, c, d) | a, b, c, \text{ and } d \text{ are positive integers with } abcd = 6\}$?

$(6,1,1,1), (1,6,1,1), (1,1,6,1), (1,1,1,6), (3,2,1,1), (3,1,2,1), (3,1,1,2), (2,3,1,1), (2,1,3,1), (2,1,1,3), (1,3,2,1), (1,3,1,2), (1,2,3,1), (1,2,1,3), (1,1,3,2),$ and $(1,1,2,3)$.

8-3

Representing Relations

8-3 Ex.8

- Determine whether the relations represented by the matrices are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

- b) $M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

- ${M_R}^{[2]} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

reflexive

~~irreflexive~~

~~symmetric~~

antisymmetric

~~transitive~~

transitive iff ${M_R}^{[2]} = M_R$

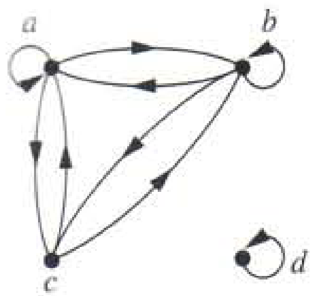
8-3 Ex.14

- Let R_1 and R_2 be relations on a set A represented by the matrices
- $M_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $M_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
- Find the matrices that represent
- c) $R_2 \circ R_1$ $\quad$ $M_{R_1} \odot M_{R_2}$
- d) $R_1 \circ R_1$ $\quad$ $M_{R_1} \odot M_{R_1}$
- e) $R_1 \oplus R_2$ $\quad$ $M_{R_1} \oplus M_{R_2}$

8-3 Ex.32

- Determine whether the relations represented by the directed graphs are reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and/or transitive.

27.



~~reflexive~~, ~~irreflexive~~, symmetric, ~~antisymmetric~~, ~~asymmetric~~, ~~transitive~~

28.

reflexive, ~~irreflexive~~, symmetric, ~~antisymmetric~~, ~~asymmetric~~, transitive

8-4

Closures of Relations

8-4 Ex.22

- Suppose that the relation R is reflexive.
- Show that R^+ is reflexive.

- Since $R \subseteq R^+$, clearly if $\Delta \subseteq R$, then $\Delta \subseteq R^+$.

8-4 Ex.26

- Find the transitive closures of the relation on $\{a, b, c, d, e\}$.
- a) $R = \{(a,c), (b,d), (c,a), (d,b), (e,d)\}$

a) We show the various matrices that are involved. First,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{A}^{[2]} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \quad \mathbf{A}^{[3]} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \mathbf{A}\,.$$

It follows that $\mathbf{A}^{[4]} = \mathbf{A}^{[2]}$ and $\mathbf{A}^{[5]} = \mathbf{A}^{[3]}$. Therefore the answer $\mathbf{B}$, the meet of all the $\mathbf{A}$'s, is $\mathbf{A} \vee \mathbf{A}^{[2]}$, namely

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

8-5

Equivalence Relations

8-5 Ex.9

- Suppose that A is a nonempty set, and f is a function that has A as its domain.
- Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$.
- a) Show that R is an equivalence relation on A.
- b) What are the equivalence classes of an integer c?
- a) Check reflexive, symmetric, transitive.
- b) The sets $\{a|f(a) = f(c)\}$

8-5 Ex.56 –c)

- Suppose that R_1 and R_2 are equivalence relations on the set S.
- Determine whether $R_1 \oplus R_2$ must be an equivalence relation.

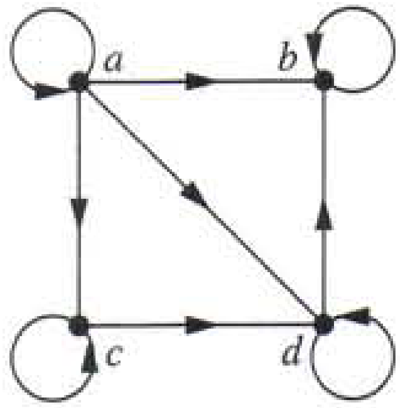
- This will never be an equivalence relation on a nonempty set, since it is not reflexive.

8-6

Partial Orderings

8-6 Ex.10

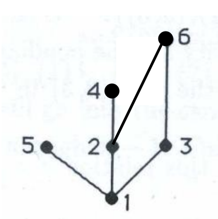
- Determine whether the relation is a partial order.

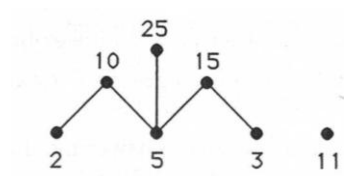


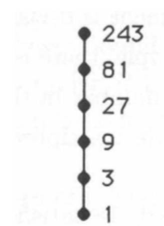
- reflexive? Y
- antisymmetric? Y
- transitive? N (no c→b)

8-6 Ex.22

- Draw the Hasse diagram for divisibility on the set
- a) {1,2,3,4,5,6}.
- b) {3,5,7,11,13,16,17}.
- c) {2,3,5,10,11,15,25}.
- d) {1,3,9,27,81,243}.

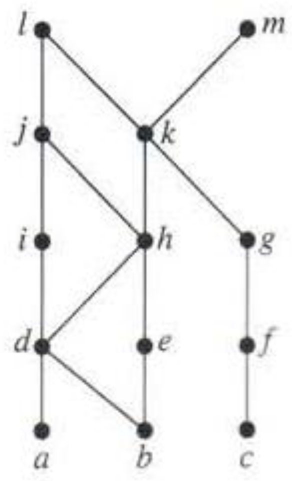






8-6 Ex.32

- Answer these questions for the partial order represented by this Hasse diagram



- a) Find the maximal elements l, m
- b) Find the minimal elements a, b, c
- c) Is there a greatest element? no
- d) Is there a least element? no
- e) Find all upper bounds of $\{a, b, c\}$ k, l, m
- f) Find the least upper bound of $\{a, b, c\}$, if exists k
- g) Find all lower bound of $\{f, g, h\}$ no
- h) Find the greatest lower bound of $\{f, g, h\}$, if exists no

8-6 Ex.44

- Determine whether these posets are lattices
  - a) $(\{1,3,6,9,12\}, |)$
  - b) $(\{1,5,25,125\}, |)$
  - c) $(\mathbf{Z}, \geq)$
  - d) $(P(S), \supseteq)$, where $P(S)$ is the power set of a set S
- N
- Y
- Y
- Y, g.l.b of two sets $A, B = A \cup B$, l.u.b $= A \cap B$.